Systems of Thought

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Systems of Thought:

Representations (Symbols) and Rules to Manipulate Them

Systems of Thought

Language

Systems of Thought

Language

Logic (Boole's *The Laws of Thought*)

Systems of Thought

Language

Logic (Boole's *The Laws of Thought*)

Mathematics

Systems of Thought

Language

Logic (Boole's *The Laws of Thought*)

Mathematics

Computation

Computational Thinking

Jeanette Wing et al.

Collection of ideas and techniques

- Algorithmic thinking
- The craft of programming
- The design of complex systems
- …

Systems of Thought

Language

Logic

Mathematics

Computation

Probabilistic reasoning, machine learning & statistics

Systems of Thought

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Probabilistic reasoning, machine learning & statistics

Visualization and visual thinking

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Probabilistic reasoning, machine learning & statistics

Visualization and visual thinking

Which one is best? (Hint: None)

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Visualization and visual thinking

Which one is best? (Hint: None)

What are the advantages of each?

Let’s Solve a Problem:

Number Scrabble

Herb Simon

Herb Simon

Nobel Prize in Economics (1977)
“for his pioneering research into the decision-making process within economic organizations”

Turing Award (1975)
"basic contributions to artificial intelligence, the psychology of human cognition, and list processing"

Number Scrabble

Goal: Pick three numbers that sum to 15

1 2 3 4 5 6 7 8 9

A:

B:

Number Scrabble

Goal: Pick three numbers that sum to 15

1 2 3 4 5 6 7 9

A: 8

B:

Number Scrabble

Goal: Pick three numbers that sum to 15

1 3 4 5 6 7 9

A: 8

B: 2

Number Scrabble

Goal: Pick three numbers that sum to 15

1 3 5 6 7 9

A: 8 4

B: 2

Number Scrabble

Goal: Pick three numbers that sum to 15

1 5 6 7 9

A: 8 4

B: 2 3

Number Scrabble

Goal: Pick three numbers that sum to 15

1 6 7 9

A: 8 4 5

B: 2 3

Number Scrabble

Goal: Pick three numbers that sum to 15

1 6 7 9

A: 8 4 5

B: 2 3 ?

Problem Isomorphs

Problem Isomorph

4	3	8
9	5	1
2	7	6

Magic Square: All rows, columns, diagonals sum to 15

Switching to a Visual Representation

4	3	8
9	5	1
2	7	6

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?

Switching to a Visual Representation

4	3	8
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2	7	6

The Representation Effect

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The representation effect: Human performance varies enormously (10-100:1) with different representations

The right representation

- Faster solution
- Fewer errors
- Better comprehension and memory
- …

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The representation effect: Human performance varies enormously (10-100:1) with different representations

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- Faster solution
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- …

But, the R. R. depends on the problem/question/task

Abstract Data Types

Choose the interface

Different possible representations of the data

Running times of key operations depend on representation

Choose the appropriate implementation for the problem at hand

Key Questions

1. What is the problem you are trying to solve?
2. How do you think about the problem? What are the semantic objects and their relationships?
3. What visual representations are already used? How does the visualization represent those objects? How does the visualization support inference?

“Number Representations”

Norman and Zhang

Number Representations

Counting – Tallying

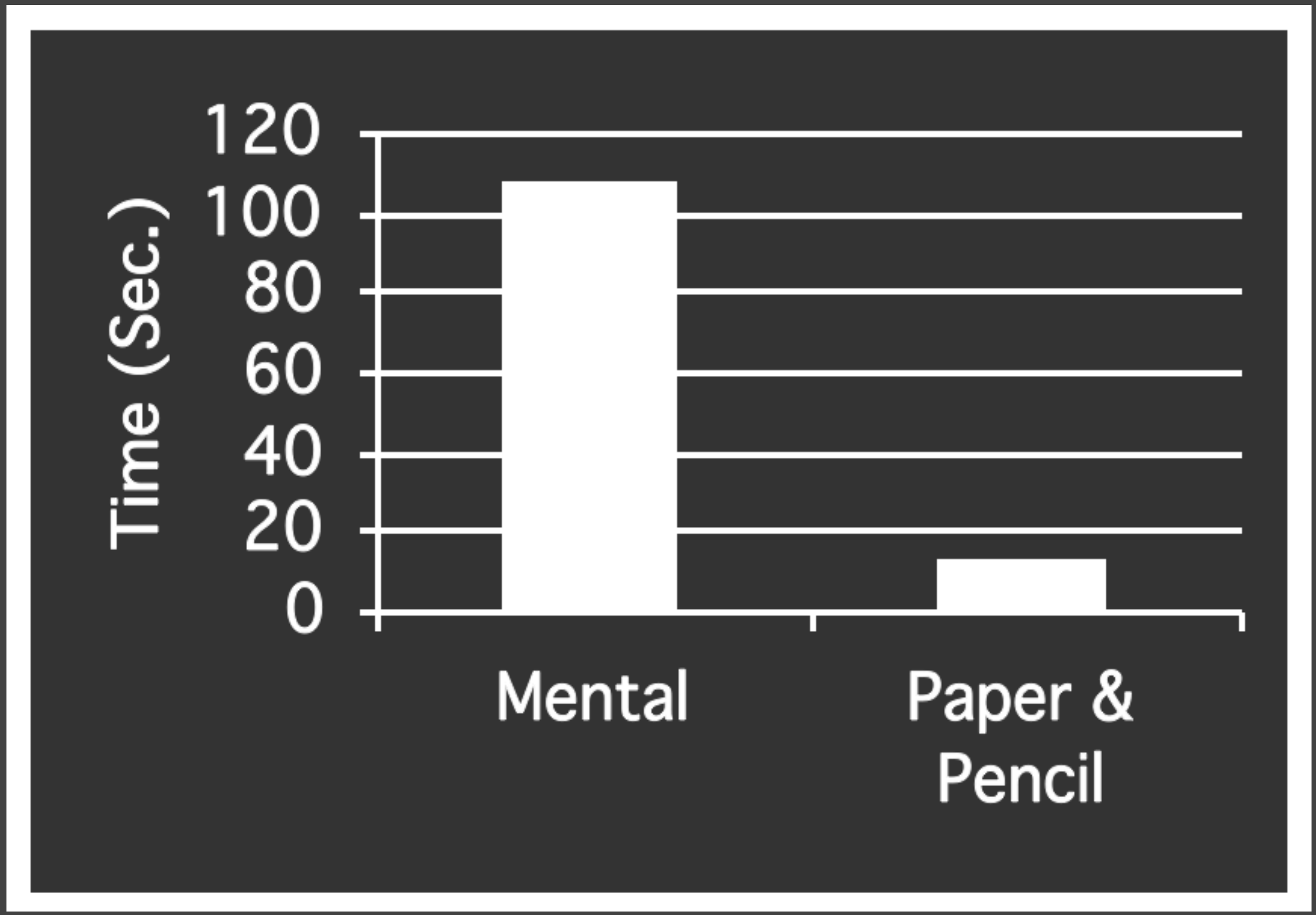
Adding – Roman numerals

XXIII + XII = XXXIIIII = XXXV

Multiplication – Arabic number systems

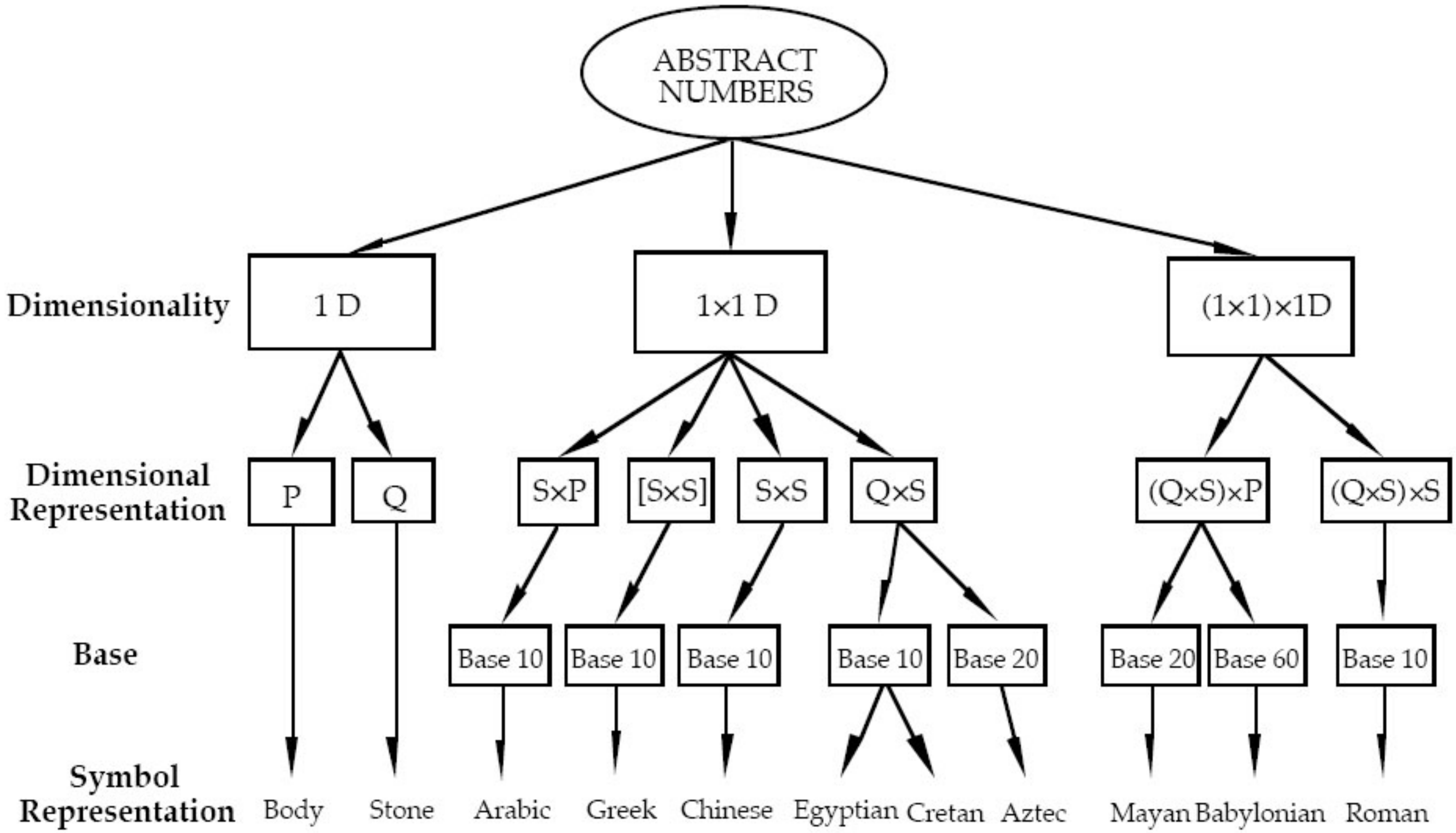
Long-Hand Multiplication

```
   34
 x 72
 ----
   68
 238
 ----
 2448
```



From "Introduction to Information Visualization,"

Card, Schneiderman, Mackinlay



Zhang and Norman, The Representations of Numbers, Cognition, 57, 271-295, 1996

Distributed Cognition

External (E) vs. Internal (I) process

		Roman	Arabic
1.	Separate power & base	I	E
2.	Get base value	E	I
3.	Multiply base values	I	I
4.	Get power values	I	E
5.	Add power values	I	E
6.	Combine base & power	I	E
7.	Add results	I	E

Arabic more efficient than Roman

Manipulation

Notation as a Tool of Thought

K. Iverson's
1979 ACM Turing Award Address

Notation as a Tool for Thought

"The thesis of the present paper is that the advantages of executabiity and universality found in programming languages can be effectively combined, in a single coherent language, with the advantages offered by mathematical notation"

K. Iverson

Arithmetic and Algebra in APL (k)

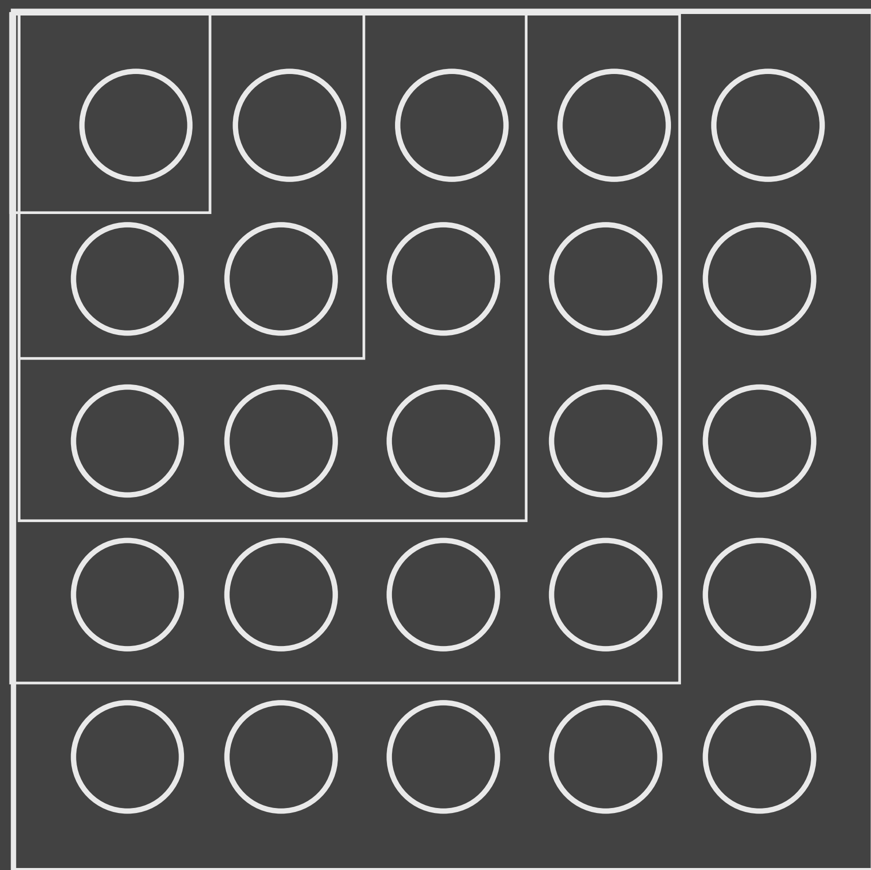
```
> k = 5
> til k
    0 1 2 3 4
> 1 + 2 * til k
    1 3 5 7 9
> +/ 1+2*til k            // 1 + 3 + 5 + 7 + 9
    25
> k*k
     25
```

Program Transformations as Proofs

```
+/ (1 + 2 * til k)                  // odd numbers
+/ (1 + (til k) + (til k))          // def of multiplication
+/ (1 + (til k) + (reverse til k))  // addition commutative
+/ (1 + k # (k-1))                  // 0 1 2 + 2 1 0 = 2 2 2
+/ k#k                              // k = k-1+1
k*k                                 // k*k = +/ k#k
                                    // 3*3 = +/ 3 3 3 = +/ 3#3
```

K. Iverson, *Arithmetic*, 1991

Visual Proofs



Algebra

$1+3+5+7+9=5^2$

http://blog.wolfram.com/2008/12/01/the-incredible-convenience-of-mathema

Key Questions

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4. How can the manipulation of the representation be used to manipulate the objects and their relationships?

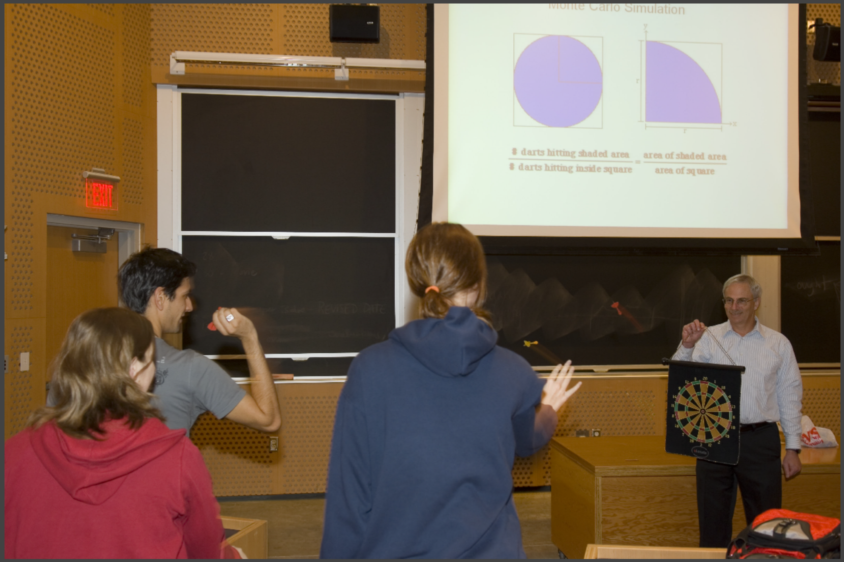
Multiple Representations

MIT Intro to Comp. Prob. Solving

Lecture 3:

- There exists a constant, π, such that π r^2 is the area of a circle
- Archimedes believed that 223/71 < π < 22/7
- The Bible later asserted that π = 3
- Who is right?

John Guttag



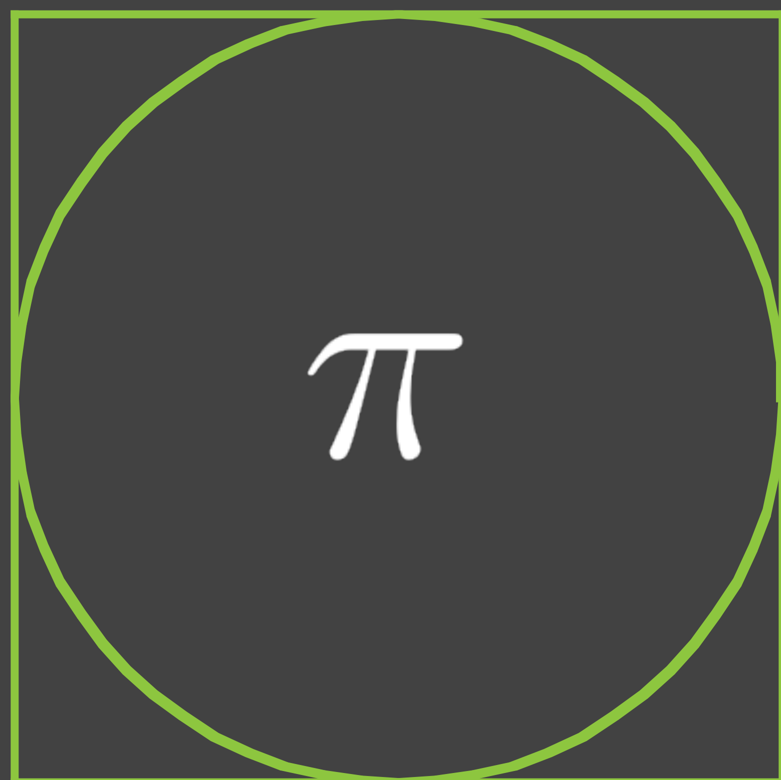
Solution:

- Buffon-Laplace simulation

Monte Carlo Solution

```
A = 0
for i in range(N):
    x = Uniform()
    y = Uniform()
    if x*x + y*y < 1:
        A += 1
p = A/N
return 4*p
```

Probability of Hitting the Dartboard



$$p = \frac{\pi}{4}$$

Therefore the area of the circle is not less than K.

Since then the area of the circle is neither greater nor less than K, it is equal to it.

Proposition 2.

The area of a circle is to the square on its diameter as 11 to 14.

[The text of this proposition is not satisfactory, and Archimedes cannot have placed it before Proposition 3, as the approximation depends upon the result of that proposition.]

Proposition 3.

The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$.

[In view of the interesting questions arising out of the arithmetical content of this proposition of Archimedes, it is necessary, in reproducing it, to distinguish carefully the actual steps set out in the text as we have it from the intermediate steps (mostly supplied by Eutocius) which it is convenient to put in for the purpose of making the proof easier to follow. Accordingly all the steps not actually appearing in the text have been enclosed in square brackets, in order that it may be clearly seen how far Archimedes omits actual calculations and only gives results. It will be observed that he gives two fractional approximations to $\sqrt{3}$ (one being less and the other greater than the real value) without any explanation as to how he arrived at them; and in like manner approximations to the square roots of several large numbers which are not complete squares are merely stated. These various approximations and the machinery of Greek arithmetic in general will be found discussed in the Introduction, Chapter IV.]

I. Let AB be the diameter of any circle, O its centre, AC the tangent at A; and let the angle AOC be one-third of a right angle.

Then $OA : AC\,[=\sqrt{3} : 1] > 265 : 153$ .... (1),

and $OC : CA\,[=2 : 1] = 306 : 153$ .... (2).

(i) First, draw OD bisecting the angle AOC and meeting AC in D. by (2)

Now $CO : OA = CD : DA$, [Eucl. VI. 3]

so that $[CO + OA : OA = CA : DA$, or]

$\frac{CO}{OA}+1 = \frac{CD+DA}{DA}$ $\quad CO + OA : CA = OA : AD$.

Therefore [by (1) and (2)]

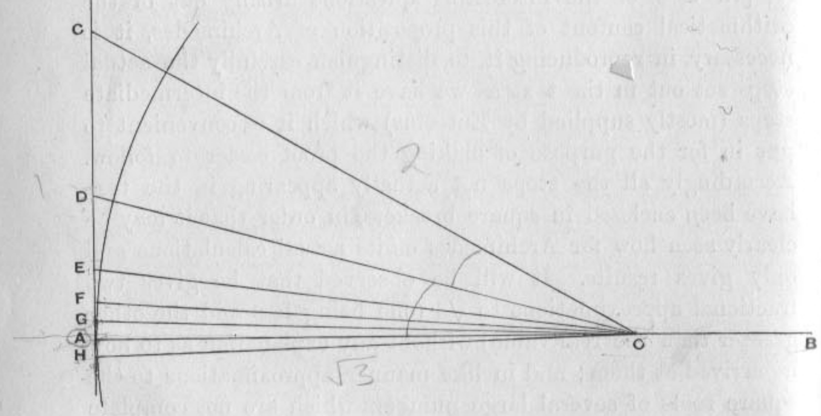
$OA : AD > 571 : 153$ .... (3).

Hence $OD^2 : AD^2\,[=(OA^2 + AD^2) : AD^2$

$\because OD^2 = OA^2 + AD^2$ $\quad > (571^2 + 153^2) : 153^2]$

$> 349450 : 23409$,

so that $OD : DA > 591\frac{1}{8} : 153$ .... (4).



(ii) Secondly, let OE bisect the angle AOD, meeting AD in E.

[Then by (?) $DO : OA = DE : EA$, $\frac{DO}{OA}+1 = \frac{DE}{EA}+1$

so that $DO + OA : DA = OA : AE.$]

Therefore $OA : AE\,[> (591\frac{1}{8} + 571) : 153$, by (3) and (4)]

$> 1162\frac{1}{8} : 153$ .... (5).

$\frac{OA}{AE} = \frac{OD}{DA} + \frac{OA}{DA} > \frac{591\frac{1}{8}}{153} + \frac{571}{153} = \frac{1162\frac{1}{8}}{153}$

by (1), (2).

[It follows that

$\frac{OA^2+OE^2}{OE^2} = \left(\frac{OA}{OE}\right)^2 + 1 > \left(\frac{1162\frac{1}{8}}{153}\right)^2 + 1$

$$OE^2 : EA^2 > \{(1162\tfrac{1}{8})^2 + 153^2\} : 153^2$$
$$> (1350534\tfrac{33}{64} + 23409) : 23409$$
$$> 1373943\tfrac{33}{64} : 23409.]$$

Thus $\qquad OE : EA > 1172\frac{1}{8} : 153$ ....(6).

(iii) *Thirdly*, let OF bisect the angle AOE and meet AE in F.

We thus obtain the result [corresponding to (3) and (5) above] that

$$OA : AF [> (1162\tfrac{1}{8} + 1172\tfrac{1}{8}) : 153]$$
$$> 2334\tfrac{1}{4} : 153 \text{ ....(7).}$$

[Therefore $\quad OF^2 : FA^2 > \{(2334\frac{1}{4})^2 + 153^2\} : 153^2$

$$> 5472132\tfrac{1}{16} : 23409.]$$

Thus $\qquad OF : FA > 2339\frac{1}{4} : 153$ ....(8).

(iv) *Fourthly*, let OG bisect the angle AOF, meeting AF in G.

We have then

$OA : AG [> (2334\frac{1}{4} + 2339\frac{1}{4}) : 153$, by means of (7) and (8)]

$$> 4673\tfrac{1}{2} : 153.$$

Now the angle AOC, which is one-third of a right angle, has been bisected four times, and it follows that

$$\angle AOG = \tfrac{1}{48} \text{ (a right angle).}$$

Make the angle AOH on the other side of OA equal to the angle AOG, and let GA produced meet OH in H.

Then $\qquad \angle GOH = \frac{1}{24}$ (a right angle).

Thus GH is one side of a regular polygon of 96 sides circumscribed to the given circle.

And, since $\qquad OA : AG > 4673\frac{1}{2} : 153$,

while $\qquad AB = 2OA, \quad GH = 2AG$,

it follows that

AB : (perimeter of polygon of 96 sides) $[> 4673\frac{1}{2} : 153 \times 96]$

$$> 4673\tfrac{1}{2} : 14688.$$

Part 1

But $\qquad \dfrac{14688}{4673\frac{1}{2}} = 3 + \dfrac{667\frac{1}{2}}{4673\frac{1}{2}}$

$$\left[< 3 + \frac{667\frac{1}{2}}{4672\frac{1}{2}}\right]$$
$$< 3\tfrac{1}{7}.$$

Therefore the circumference of the circle (being less than the perimeter of the polygon) is *a fortiori* less than $3\frac{1}{7}$ times the diameter AB.

II. Next let AB be the diameter of a circle, and let AC, meeting the circle in C, make the angle CAB equal to one-third of a right angle. Join BC.

Then $\qquad AC : CB [= \sqrt{3} : 1] < 1351 : 780.$

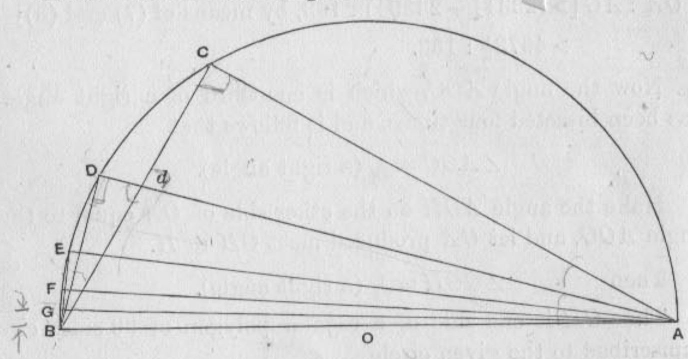
step (i) *First*, let AD bisect the angle BAC and meet BC in d and the circle in D. Join BD.

Then $\qquad \angle BAD = \angle dAC$

$$= \angle dBD,$$

and the angles at D, C are both right angles.

It follows that the triangles ADB, $[ACd]$, BDd are similar.



Therefore $\qquad AD : DB = BD : Dd$

$$[= AC : Cd]$$
$$= AB : Bd \qquad \text{[Eucl. VI. 3]}$$
$$= AB + AC : Bd + Cd$$
$$= AB + AC : BC$$

or $\qquad BA + AC : BC = AD : DB.$

$\frac{AD}{DB} = \frac{BA+AC}{BC} = \frac{BA}{BC} + \frac{AC}{BC} < 2 + \frac{1351}{780}$

[But $\quad AC : CB < 1351 : 780$, from above,

while $\quad BA : BC = 2 : 1$

$\qquad\qquad = 1560 : 780.]$

$\frac{AB^2}{BD^2} = \frac{AD^2+BD^2}{BD^2} = \left(\frac{AD}{BD}\right)^2 + 1$

Therefore $\quad AD : DB < 2911 : 780$ ....(1).

[Hence $\quad AB^2 : BD^2 < (2911^2 + 780^2) : 780^2$

$AD^2 + DB^2$

$\qquad\qquad < 9082321 : 608400.]$

Thus $\quad AB : BD < 3013\frac{3}{4} : 780$ ....(2).

step (i)

Secondly, let AE bisect the angle BAD, meeting the circle in E; and let BE be joined.

Then we prove, in the same way as before, that

$$AE : EB [= BA + AD : BD$$
$$< (3013\tfrac{3}{4} + 2911) : 780, \text{ by (1) and (2)}]$$
$$< 5924\tfrac{3}{4} : 780$$
$$< 5924\tfrac{3}{4} \times \tfrac{4}{13} : 780 \times \tfrac{4}{13}$$
$$< 1823 : 240 \text{ ....(3).}$$

[Hence $\quad AB^2 : BE^2 < (1823^2 + 240^2) : 240^2$

$\qquad\qquad < 3380929 : 57600.]$

Therefore $\quad AB : BE < 1838\frac{9}{11} : 240$..................(4).

Thirdly, let AF bisect the angle BAE, meeting the circle in F.

Thus $\quad AF : FB [= BA + AE : BE$

$$< 3661\tfrac{9}{11} : 240, \text{ by (3) and (4)}]$$
$$< 3661\tfrac{9}{11} \times \tfrac{11}{40} : 240 \times \tfrac{11}{40}$$
$$< 1007 : 66 \text{ ....(5).}$$

[It follows that

$$AB^2 : BF^2 < (1007^2 + 66^2) : 66^2$$
$$< 1018405 : 4356.]$$

Therefore $\quad AB : BF < 1009\frac{1}{6} : 66$....................(6).

Fourthly, let the angle BAF be bisected by AG meeting the circle in G.

Then $\quad AG : GB [= BA + AF : BF]$

$$< 2016\tfrac{1}{6} : 66, \text{ by (5) and (6).}$$

[And $\quad AB^2 : BG^2 < \{(2016\frac{1}{6})^2 + 66^2\} : 66^2$

$\qquad\qquad < 4069284\frac{1}{36} : 4356.]$

Therefore $\quad AB : BG < 2017\frac{1}{4} : 66$,

whence $\quad BG : AB > 66 : 2017\frac{1}{4}$........................(7).

[Now the angle BAG which is the result of the fourth bisection of the angle BAC, or of one-third of a right angle, is equal to one-fortyeighth of a right angle.

Thus the angle subtended by BG at the centre is

$\frac{1}{24}$ (a right angle).]

Therefore BG is a side of a regular inscribed polygon of 96 sides.

It follows from (7) that

(perimeter of polygon) : AB [$> 96 \times 66 : 2017\frac{1}{4}$]

$\qquad\qquad > 6336 : 2017\frac{1}{4}$.

And $\quad \dfrac{6336}{2017\frac{1}{4}} > 3\frac{10}{71}$.

Much more then is the circumference of the circle greater than $3\frac{10}{71}$ times the diameter.

Thus the ratio of the circumference to the diameter

$$< 3\tfrac{1}{7} \text{ but } > 3\tfrac{10}{71}.$$

Archimedes Traps Pi

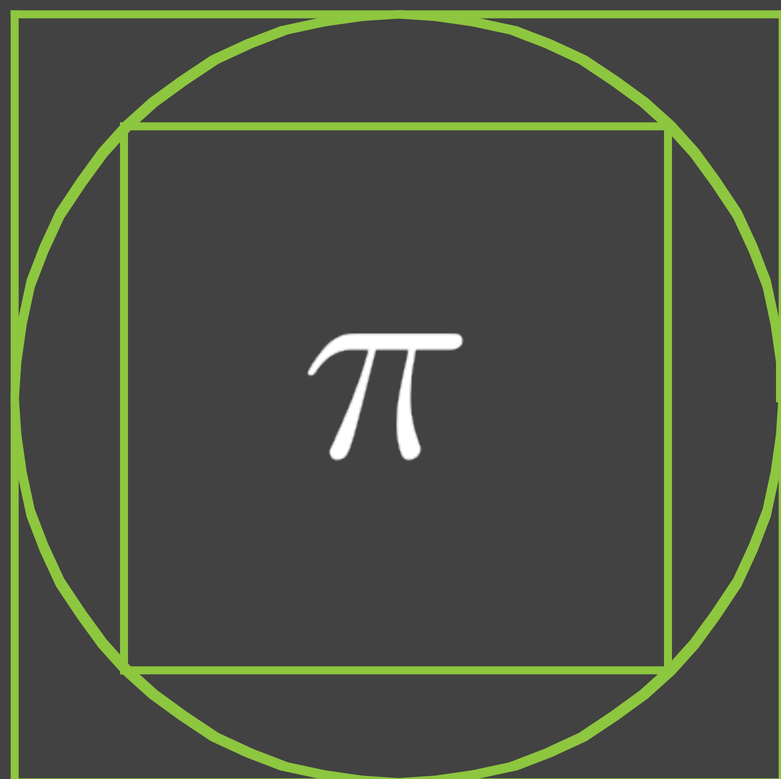
Archimedes Traps Pi

4 sided polygon

$$\pi < 4$$

Archimedes Traps Pi

4 sided polygon



$$\frac{1}{2} < \pi < 4$$

Archimedes Traps Pi

96 sided polygon

$$\frac{22}{7} < \pi < \frac{223}{71}$$

Monte Carlo Solution

```
A = 0
for i in range(N):
    x = Uniform()
    y = Uniform()
    if x*x + y*y < 1:
        A += 1
p = A/N
return 4*p
```

Mathematics of Monte Carlo

$$f(x, y) = x^2 + y^2 < 1$$

$$Z = f(X, Y)$$

$$E[Z] = \frac{1}{N} \sum z_i$$

Central limit theorem for Bernoulli process

$$\Pr(E[Z] > z) = N\left(\frac{E(Z) - z}{\sqrt{Np(1-p)}}\right)$$

Key Questions

1. What is the problem you are trying to solve?
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3. What visual representations are already used? How does the visualization represent those objects and support reasoning about them?
4. How can the manipulation of the representation be used to manipulate the objects and their relationships?
5. How can visualization be coupled with other systems of thought?

Thank you

Questions?